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A MODIFIED QUADRATIC COST PROBLEM AND FEEDBACK STABILIZATION OF--ETC(U)

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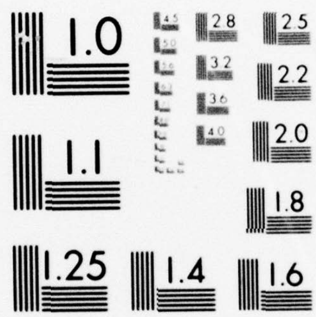

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A MODIFIED QUADRATIC COST PROBLEM AND FEEDBACK STABILIZATION OF LINEAR DISCRETE TIME SYSTEMS

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A Modified Quadratic Cost Problem and Feedback Stabilization of Linear Discrete Time Systems*

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Abstract

This report considers a feedback control law for linear time-varying and time invariant discrete systems based on a receding horizon concept applied to a minimum energy problem with fixed terminal constraints. The control law is shown to be asymptotically stable and to result in a new method for stabilizing linear time-varying systems as well as extending some well known methods for stabilizing time invariant systems. In particular, the stabilizing gains of the feedback control law are obtained from the solution to a discrete Riccati equation over an arbitrary finite time interval, which is relatively easy to compute. The gain matrix reduces to a constant matrix for linear time invariant systems. Some stability results in [2] and [5] will turn out to be special cases of these results. The results parallel those of [4] for linear continuous time systems, although the technical details are tedious and more involved.

I. Introduction

Consider a linear discrete system

$$x(i+1) = \Phi_i \, x(i) + B(i)u(i) \tag{1.1}$$

$$y(i) = C(i)x(i) \tag{1.2}$$

where Φ_i, $B(i)$ and $C(i)$ are nxn, nxn and pxn matrices, and Φ_i is assumed to be nonsingular.* Consider also a cost function

$$\sum_{i=i_o}^{i_f-1} y'(i)Q(i)y(i) + u'(i)R(i)u(i) \tag{1.3}$$

where $Q(i) = D'(i)D(i) \geq 0$ and $R(i) > 0$, together with the boundary conditions

$$\begin{aligned} x(i_o) &= x_o \\ x(i_f) &= 0. \end{aligned} \tag{1.4}$$

The optimal solution is obtained by introducing the 2n-dimensional Hamiltonian system [11]

$$\begin{bmatrix} x(i+1) \\ p(i) \end{bmatrix} = \begin{bmatrix} \Phi_i, & -B(i)R^{-1}(i)B'(i) \\ C'(i)Q(i)C(i), & \Phi_i' \end{bmatrix} \begin{bmatrix} x(i) \\ p(i+1) \end{bmatrix} \tag{1.5}$$

with the optimal control

$$u(i) = -R^{-1}(i)B'(i)p(i+1) \, . \tag{1.6}$$

The equivalent representation of (1.5) is given by

$$\begin{bmatrix} x(i+1) \\ p(i+1) \end{bmatrix} = \begin{bmatrix} \Phi_i + B(i)R^{-1}(i)B'(i)\Phi_i'^{-1}C'(i)Q(i)C(i), & -B(i)R^{-1}(i)B'(i)\Phi_i'^{-1} \\ -\Phi_i'^{-1}C'(i)Q(i)C(i), & \Phi_i'^{-1} \end{bmatrix} \begin{bmatrix} x(i) \\ p(i) \end{bmatrix} \tag{1.7}$$

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*The shorthand notation $\Phi_i = \Phi(i+1,i)$ will be used for the state transition matrix when applicable.

Let $S(i,i_o)$ denote the $2n \times 2n$ state transition matrix of the system (1.7) with the partitioning into four $n \times n$ submatrices:

$$S(i,i_o) = \begin{bmatrix} \psi(i,i_o) & \Omega(i,i_o) \\ \chi(i,i_o) & \Lambda(i,i_o) \end{bmatrix} . \tag{1.8}$$

The open-loop optimal control is given by

$$u(i) = -R^{-1}(i)B'(i)\left[\chi(i+1,i_o)-\Lambda(i+1,i_o)\Omega^{-1}(i_f,i_o)\psi(i_f,i_o)\right]x(i_o) , \tag{1.9}$$

when the inverse in (1.9) exists. The optimal closed loop control is given by

$$u(i) = R^{-1}(i)B'(i)\,\Omega^{-1}(i_f,i+1)\psi(i_f,i+1)x(i+1) \tag{1.10}$$

whenever the inverse in (1.10) exists. Another representation of (1.10) will be given in Section III.

The standard regulator problem, which minimizes the cost function (1.3) with a free terminal condition, has the solution given by

$$u(i) = -R^{-1}(i)B'(i)\left[I+K(i+1,i_f)B(i)R^{-1}(i)B'(i)\right]^{-1}K(i+1,i_f)\Phi_i x(i) , \tag{1.11}$$

where $K(i,j)$ is obtained from

$$K(i,j) = \Phi_i' K(i+1,j)\Phi_i - \Phi_i' K(i+1,j)B(i)\left[R(i) + B'(i)K(i+1,j)B(i)\right]^{-1} \times B'(i)K(i+1,j)\Phi_i + C'(i)Q(i)C(i), \qquad K(j,j) = 0 . \tag{1.12}$$

The following definition is necessary for further analyses.

<u>Definition</u>. The pair $\{\Phi(i+1,i) , B(i)\}$ is said to be uniformly completely controllable if for some positive integer $\ell_c \geq 1$ the following conditions hold:

(1) $\alpha_1 I \leq W(i,i+\ell_c) \leq \alpha_2 I$ for all i (1.13)

(2) $||\Phi(i,j)|| \leq \alpha_3 (|i-j|)$ for all i,j * , (1.14)

* Throughout the paper the Euclidean norm is assumed for vectors and the spectral norm induced by the Euclidean norm for matrices.

where the controllability matrix $W(i,j)$ is defined by

$$W(i,j) = \sum_{k=i}^{j-1} \Phi(i,k+1)B(k)B'(k)\Phi'(i,k+1) \quad , \tag{1.15}$$

α_1 and α_2 are positive constants, and $\alpha_3(.)$ maps $\mathbb{R}$ into $\mathbb{R}$ and is bounded on bounded intervals.

The uniform complete observability of the pair $\{\Phi(i+1,i) , C(i)\}$ is defined similarly as above with the observability matrix

$$M(i,j) = \sum_{k=i}^{j-1} \Phi'(k,i)C'(k)C(k)\Phi(k,i) \tag{1.16}$$

and with a positive integer $\ell_o \geq 1$. Let $\ell = \max\{\ell_c,\ell_o\}$. It is well known [1,2] that under uniform complete controllability and observability conditions together with

$$\alpha_4 I \leq Q(i) \leq \alpha_5 I \text{ and } \alpha_6 I \leq R(i) \leq \alpha_7 I \quad , \tag{1.17}$$

where α_4 , α_5 , α_6 and α_7 are positive constants, the steady state control law (1.11) with $i_f = \infty$ is uniformly asymptotically stable, but practically speaking, it is very difficult to compute $K(i,\infty)$ for the stable control. We will show that a modification of the control (1.10) results in an asymptotically stable control and is optimal in a certain sense. In particular, the gain matrix for the new control is obtained by solving a Riccati equation on an arbitrary finite time interval (larger than ℓ_c) , which is relatively easy to compute in relation to the infinite time interval for (1.11) with $i_f = \infty$. Preliminaries and some basic results are given in Section II. The results for time-varying systems are discussed in Section III and time invariant systems in Section IV. A dual problem will be discussed in Section V. Throughout this paper the following matrix identity is used frequently

$$I - Y'(ZY' + X)^{-1} Z = (I + Y'X^{-1}Z)^{-1} \tag{1.18}$$

whenever the inverses exist.

II. Some Basic Results.

In this section some basic results are given which are necessary for the succeeding sections. The corresponding results for continuous-time systems are well known [6,7 and 8]. It appears, however, that a similar treatment of discrete systems is not available in the control literature. Thus we sketch the proofs briefly in the Appendix.

Theorem 2.1. The solution of the matrix Riccati equation (1.12) is bounded below by

$$K(i_o, i_f) \geq \frac{1}{1 + \|G\|} N(i_o, i_f) \; , \tag{2.1}$$

where $N(i,j)$ is the solution of the matrix Lyapunov equation

$$\begin{aligned} N(i,j) &= \Phi_i' N(i+1,j)\Phi_i + C'(i)Q(i)C(i) \\ N(j,j) &= 0 \end{aligned} \tag{2.2}$$

and the operator G on $\ell_2([i_o, i_f - 1] \, , \mathbb{R}^p)$ is given by

$$(Gu)(i) = \sum_{k=i_o}^{i-1} C(i)\Phi(i,k+1)B(k)u(k) \quad . \tag{2.3}$$

Proof. See Appendix A.

As a consequence of uniform complete controllability, there exist positive constants α_8 , α_9 and α_{10} such that

$$\begin{aligned} \|\Phi(i+1,i)\| &\leq \alpha_8 \\ \|\Phi^{-1}(i+1,i)\| &\leq \alpha_9 \\ \|B(i)\| &\leq \alpha_{10} \end{aligned} \tag{2.4}$$

for all i . The positive constants α_8 and α_9 are given by $\alpha_8 = \alpha_3(1)$ and $\alpha_9 = \alpha_3(1)$ from (1.14). From (1.13) - (1.15) we have

$$\Phi(i,k+1)B(k)B'(k)\Phi'(i,k+1) \le \alpha_2 I, \; k\varepsilon\, [i, i+\ell_c - 1].$$

From the above inequality it follows that

$$\begin{aligned} n\alpha_2 &\ge \operatorname{tr} \Phi(i,k+1)B(k)B'(k)\Phi'(i,k+1) \\ &= \operatorname{tr} B(k)B'(k)\Phi'(i,k+1)\Phi(i,k+1) \\ &\ge \operatorname{tr} B(k)B'(k)\lambda_{min}\, [\Phi'(i,k+1)\Phi(i,k+1)] \\ &= \operatorname{tr} B(k)B'(k)\lambda^{-1}_{max}\, [\Phi(k+1,i)\Phi'(k+1,i)] \\ &\ge \operatorname{tr} B(k)B'(k)[\max_{1\le k\le \ell_c} \alpha_3(k)]^{-2} . \end{aligned}$$

Thus we have

$$||B(k)|| = ||B'(k)|| \le [\operatorname{tr} B(k)B'(k)]^{1/2} \le [\max_{1\le k\le \ell_c} \alpha_3(k)]\, \sqrt{n\alpha_2} \triangleq \alpha_{10} .$$

The invariance of the uniform controllability of the system (1.1) under state feedback control is stated in the next theorem.

Theorem 2.2 . The uniform complete controllability of the system (1.1) is invariant under a state feedback control of the form

$$u(i) = K(i)x(i) + v(i) \tag{2.5}$$

provided $||K(i)|| \le \alpha_{12}$ for some positive constant α_{12} .

Proof: See Appendix B.

III. Linear Time-Varying Systems.

For linear time-varying systems, there exist few general methods to stabilize the linear system (1.1), one of which is the steady state control for (1.11) as mentioned before. We will suggest another feedback control law based on a receding horizon concept which stabilized (1.1) and is optimal in a certain sense. First of all, we represent the fixed terminal control law (1.10) in terms of a Riccati equation.

<u>Theorem 3.1</u>. The fixed terminal optimal closed-loop control law (1.10) can be represented as

$$u(i) = -R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i_f+1)\Phi_i x(i) \ , \tag{3.1}$$

if the inverse in (3.1) exists, where $\hat{P}(i,j)$ satisfies

$$\begin{aligned}\hat{P}(i,j) = {} & \Phi_i^{-1}\hat{P}(i+1,j)\Phi_i'^{-1} - \Phi_i^{-1}\hat{P}(i+1,j)\Phi_i'^{-1}C'(i)D'(i)[I \\ & + D(i)C(i)\Phi_i^{-1}\hat{P}(i+1,j)\Phi_i'^{-1}C'(i)D'(i)]^{-1}D(i)C(i)\Phi_i^{-1}\hat{P}(i+1,j)\Phi_i'^{-1} \\ & + B(i-1)R^{-1}(i-1)B'(i-1) \ , \quad \hat{P}(j,j) = 0 \ .\end{aligned} \tag{3.2}$$

Proof: We can obtain the relation (3.2) in a few equivalent ways. It can be obtained by letting $K(i,j) \triangleq P^{-1}(i,j)$ in Equation (1.12) with $K(j,j) = \infty$. Thus we have

$$\begin{aligned}P(i,j) = {} & \{\Phi_i'P^{-1}(i+1,j)\Phi_i - \Phi_i'P^{-1}(i+1,j)B(i)[R(i) + \\ & B'(i)P^{-1}(i+1,j)B(i)]^{-1}B'(i)P^{-1}(i+1,j)\Phi_i + \\ & C'(i)D'(i)D(i)C(i)\}^{-1} \\ = {} & \Phi_i^{-1}\{P^{-1}(i+i,j)[I-B(i)(R(i)+B'(i)P^{-1}(i+1,j)B(i))B'(i)P^{-1}(i+1,j)] + \\ & \Phi_i'^{-1}C'(i)D'(i)D(i)C(i)\Phi_i^{-1}\}^{-1}\Phi_i'^{-1} \\ = {} & \Phi_i^{-1}\{P^{-1}(i+1,j)[I+B(i)R^{-1}(i)B'(i)P^{-1}(i+1,j)]^{-1} + \\ & \Phi_i'^{-1}C'(i)D'(i)D(i)C(i)\Phi_i^{-1}\}^{-1}\Phi_i'^{-1} \\ = {} & \Phi_i^{-1}[P(i+1,j)+B(i)R^{-1}(i)B'(i)]\{I+\Phi_i'^{-1}C'(i)D'(i)D(i)C(i) \\ \times {} & \Phi_i^{-1}[P(i+1,j)+B(i)R^{-1}(i)B'(i)]\}^{-1}\Phi_i'^{-1} \ .\end{aligned}$$

Let

$$\hat{P}(i,j) \triangleq P(i,j) + B(i-1)R^{-1}(i-1)B'(i-1) \quad . \tag{3.3}$$

Then the above equation can be expressed as

$$\hat{P}(i,j) = \Phi_i^{-1}\hat{P}(i+1,j)\left\{I + \Phi_i'^{-1} C'(i)D'(i)D(i)C(i)\Phi_i^{-1}\hat{P}(i+1,j)\right\}^{-1} \Phi_i'^{-1}$$
$$+ B(i-1)R^{-1}(i-1)B'(i-1) \quad . \tag{3.4}$$

Combining (3.4) and (1.18) yields (3.2). Since $P(i_f,i_f) = 0$, $\hat{P}(i_f,i_f) = B(i_f-1)R^{-1}(i_f-1)B'(i_f-1)$. This is equivalent to $\hat{P}(i_f+1,i_f+1) = 0$ from (3.2). Combining (1.11) and (3.3) yields (3.1). This completes the proof.

It is noted that the optimal cost of the system (1.1) with the control (3.1) is given by

$$x'(i_o)P^{-1}(i_o,i_f)x(i_o) \quad . \tag{3.5}$$

The following lemma is necessary for the main theorems.

Lemma 3.1.

(1) $\hat{P}(i,j_1) \leq \hat{P}(i,j_2)$ for $i \leq j_1 \leq j_2$ (3.6)

(2) $P^{-1}(i,j_1) \geq P^{-1}(i,j_2)$ for $i \leq j_1 \leq j_2$ (3.7)

(3) Assume $R(i)$ satisfies (1.17) and $0 \leq Q(i) \leq \alpha_5 I$. If the pair $\left\{\Phi(i+1,i) , B(i)\right\}$ is uniformly completely controllable and $C(i)$ is bounded such that $||C(i)|| \leq \alpha_{13}$ for all i , then for a fixed N satisfying $\ell_c \leq N < \infty$ there exist positive constants α_{14} and α_{15} such that

$$\alpha_{14} I \leq \hat{P}(i,i+N) \leq \alpha_{15} I \quad . \tag{3.8}$$

(4) Assume $R(i)$ and $Q(i)$ satisfy (1.17). If the pairs $\{\Phi(i+1,i), B(i)\}$ and $\{\Phi(i+1,i) , C(i)\}$ are uniformly completely controllable and observable respectively, then for a fixed N satisfying $\ell \leq N \leq \infty$ there exist positive constants α_{16} and α_{17} such that

$$\alpha_{16} I \leq \hat{P}(i,i+N) \leq \alpha_{17} I \quad . \tag{3.9}$$

Proof: See Appendix C.

From a receding horizon concept a new control law is obtained by replacing i_f by $i+N$ in (3.1): thus,

$$u(i) = -R^{-1}(i)B'(i)[B(i)R^{-1}(i)B'(i)+P(i+1,i+1+N)]^{-1}\Phi_i x(i) \tag{3.10}$$

$$= -R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)\Phi_i x(i) \ , \ N \geq \ell_c \tag{3.11}$$

where $\hat{P}(i,j)$ may be obtained from (3.2). Some characteristics of $\hat{P}(i,j)$ are illustrated in Fig. 1. The matrix $\hat{P}(i+1,i+1+N)$ is obtained by summing (3.2) backward from $i+1+N$ to $i+1$ on a finite time interval. The most important property of the control law (3.11) is that it is a stable control, though it is obtained from a Riccati equation on a finite time interval.

Theorem 3.2.

(1). Assume $R(i)$ satisfies (1.17) and $0 \leq Q(i) \leq \alpha_5 I$. If the pair $\{\Phi(i+1,i),B(i)\}$ is uniformly completely controllable and $C(i)$ is bounded such that $||C(i)|| \leq \alpha_{13}$ for all i , then for a fixed N satisfying $\ell_c+1 \leq N < \infty$, the system (1.1) - (1.2) with the feedback control law (3.11) is uniformly asymptotically stable. (Note: $Q(i)$ and $C(i)$ can be identically zero).

(2). Assume $R(i)$ and $Q(i)$ satisfy (1.17). If the pairs $\{\Phi(i+1,i),B(i)\}$ and $\{\Phi(i+1,i),C(i)\}$ are uniformly completely controllable and observable respectively, then for a fixed N satisfying $\ell + 1 \leq N \leq \infty$ the system (1.1) -(1.2) with the feedback control law (3.11) is uniformly asymptotically stable.

Proof: Consider the adjoint system of (1.1)-(1.2) with control law (3.11):

$$\hat{x}(i+1) = [\Phi_i - B(i)R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)\Phi_i]'^{-1}\hat{x}(i) \tag{3.12}$$

together with the associated scalar valued function

$$V(\hat{x},i) = \hat{x}'\Phi_i^{-1}\hat{P}(i+1,i+1+N)\Phi_i'^{-1}\hat{x} \quad . \tag{3.13}$$

From Lemma 3.1 (1) and (2.4), $V(\hat{x},i)$ satisfies

$$\alpha_{14}\alpha_9^{-2}\,|\hat{x}|^2 \leq V(\hat{x},i) \leq \alpha_{15}\alpha_9^{2}|\hat{x}|^2 \tag{3.14}$$

under the conditions of part (1). A similar inequality involving α_{16} and α_{17} can be obtained under the conditions of part (2). Thus $V(\hat{x},i)$ is a positive definite function of $\hat{x}$ under either set of conditions. The difference of (3.13) along the solution of the adjoint system (3.12) is given as follows:

$$\begin{aligned}
&V(\hat{x}(i),i) - V(\hat{x}(i+1),i+1)\\
&\quad = \hat{x}'(i)\Phi_i^{-1}\hat{P}(i+1,i+1+N)\Phi_i'^{-1}\hat{x}(i)\\
&\quad - \hat{x}'(i+1)\Phi_{i+1}^{-1}\hat{P}(i+2,i+2+N)\Phi_{i+1}'^{-1}\hat{x}(i+1)\\
&\quad = \hat{x}'(i+1)[I - B(i)R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)]\hat{P}(i+1,i+1+N)[I - \hat{P}^{-1}(i+1,i+1+N)\\
&\quad \times B(i)R^{-1}(i)B'(i)]\hat{x}(i+1) - \hat{x}'(i+1)\Phi_{i+1}^{-1}\hat{P}(i+2,i+2+N)\Phi_{i+1}'^{-1}\hat{x}(i+1)\\
&\quad = -\hat{x}'(i+1)[B(i)R^{-1}(i)B'(i) - B(i)R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)B(i)R^{-1}(i)\\
&\quad \times B'(i)]\hat{x}(i+1) - \hat{x}'(i+1)[B(i)R^{-1}(i)B'(i) - \hat{P}(i+1,i+1+N)\\
&\quad + \Phi_{i+1}^{-1}\hat{P}(i+2,i+2+N)\Phi_{i+1}'^{-1}]\hat{x}(i+1) \quad .
\end{aligned} \tag{3.15}$$

From (3.2) we have

$$\hat{P}(i+1,i+2+N) = \Phi_{i+1}^{-1}\hat{P}(i+2,i+2+N)\Phi_{i+1}'^{-1} + B(i)R^{-1}(i)B'(i) - Z(i) \ , \tag{3.16}$$

where $Z(i)$ is the non-negative definite second term on the right side of (3.2) with i and j replaced by i+1 and i+2+N respectively.

From (3.16) it follows that

$$B(i)R^{-1}(i)B'(i) - \hat{P}(i+1,i+1+N) + \Phi_{i+1}^{-1}\ \hat{P}(i+2,i+2+N)\Phi_{i+1}'^{-1}$$

$$= \hat{P}(i+1,i+2+N) - \hat{P}(i+1,i+1+N) + Z(i) \geq 0 , \qquad (3.17)$$

where the last inequality in (3.17) follows from Lemma 3.1(1). Thus the relation (3.15) can be expressed as

$$V(\hat{x}(i),i), - V(\hat{x}(i+1),i+1)$$

$$\leq -\hat{x}'(i+1)[B(i)R^{-1}(i)B'(i)-B(i)R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)B(i)R^{-1}(i)B'(i)]\hat{x}(i+1)$$

$$= -\hat{x}'(i+1)B(i)R^{-1/2}(i)S(i)R^{-1/2}(i)B'(i)\hat{x}(i+1) \quad ,$$

where $S(i)$ is defined by

$$S(i) \triangleq I - R^{-1/2}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)B(i)R^{-1/2}(i) \leq I \ . \qquad (3.18)$$

If we can show that $S(i) \geq \alpha_{18}I$ for some positive constant α_{18} and for all i, then the adjoint system (3.12) will be asymptotically unstable which is possible if and only if the original system is asymptotically stable. The proof proceeds as follows:

$$V(\hat{x}(i),i) - V(\hat{x}(i+1),i+1) \leq - \alpha_{18}\alpha_6\hat{x}'(i+1)B(i)B'(i)\hat{x}(i+1) \quad .$$

Thus we can have

$$V(\hat{x}(i+1;\hat{x}_o,i_o),i+1) - V(\hat{x}_o,i_o)$$

$$\geq \alpha_{18}\alpha_6\ \hat{x}_o' \left[\sum_{k=i_o}^{i} \Phi_p(i_o,k+1)B(k)B'(k)\Phi_p'(i,k+1) \right] \hat{x}_o$$

$$\geq \alpha_{18}\alpha_6\alpha_{19}\ |\hat{x}_o|^2 \quad \text{for some } \alpha_{19} > 0 \text{ and for } i \geq i_o + \ell_c, \qquad (3.19)$$

where $\Phi_p(i,i_o)$ is the state transition matrix of the closed-loop system

(3.1)-(3.11). The last inequality in (3.19) follows from Theorem 2.2 in that the pair $\{\Phi_i - B(i)R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)\Phi_i \;, \; B(i)\}$ is uniformly completely controllable since $R^{-1}(i)B'(i)\hat{P}^{-1}(i+1,i+1+N)\Phi_i$ is bounded from (1.17), (3.8), (3.9) and (2.4). Therefore, the solutions of the adjoint system (3.12) can be shown to increase exponentially. In turn, this can be shown to imply that the system (1.1)-(3.11) is uniformly asymptotically stable. Thus, it remains to check the lower bound of the matrix S(i) defined in (3.18). From (3.2) we have

$$S(i) \stackrel{\Delta}{=} I - R^{-1/2}(i)B'(i)[G(i) + B(i)R^{-1}(i)B(i)]^{-1}B(i)R^{-1/2}$$

$$= \left[I + R^{-1/2}(i)B'(i)G^{-1}(i)B(i)R^{-1/2} \right]^{-1} \;, \qquad (3.20)$$

where

$$G(i) \stackrel{\Delta}{=} \Phi_{i+1}^{-1}\hat{P}(i+2,\; i+1+N)\{I - \Phi'^{-1}_{i+1}C'(i+1)D'(i+1)[I + D(i+1)C(i+1)\Phi_{i+1}^{-1}\hat{P}(i+2,i+1+N)$$

$$\times\Phi'^{-1}_{i+1}C'(i+1)D'(i+1)]^{-1}D(i+1)C(i+1)\Phi_{i+1}^{-1}\hat{P}(i+2,i+1+N)\}\Phi'^{-1}_{i+1}$$

$$= \Phi_{i+1}^{-1}\hat{P}^{1/2}(i+2,i+1+N)H(i)\hat{P}^{1/2}(i+2,i+1+N)\Phi'^{-1}_{i+1} \qquad (3.21)$$

and where

$$H(i) \stackrel{\Delta}{=} I - \hat{P}^{1/2}(i+2,i+1+N)\Phi'^{-1}_{i+1}C'(i+1)D'(i+1)[I+D(i+1)C(i+1)\Phi_{i+1}^{-1}\hat{P}(i+2,i+1+N)$$

$$\times\; \Phi'^{-1}_{i+1}C'(i+1)D'(i+1)]^{-1}D(i+1)C(i+1)\Phi_{i+1}^{-1}\hat{P}^{1/2}(i+2,\; i+1+N)$$

$$= [I + \hat{P}^{1/2}(i+2,\; i+1+N)\Phi'^{-1}_{i+1}\; C'(i+1)D'(i+1)D(i+1)C(i+1)\Phi_{i+1}^{-1}\hat{P}^{1/2}(i+2,i+1+N)]^{-1} \qquad (3.22)$$

Note that G(i) is nonsingular for $N \geq \ell_c + 1$.

From (3.20), (3.21) and (3.22) it follows that

$$\| S(i) \| \geq [1+\alpha_6^{-1}\alpha_{10}^2\alpha_8^{-1}\alpha_{14}(1+\alpha_5\alpha_{13}^2\alpha_9^2\alpha_{15})^{-1}]^{-1} \stackrel{\Delta}{=} \alpha_{18} \;.$$

This completes the proof.

After careful inspection of the matrix Riccati equation (3.2) and the control law (3.11), it can be deduced that the control law (3.11) is the optimal control law for the system (1.1)-(1.2) which minimizes the moving cost function

$$\sum_{k=i}^{i+N-1} y'(k)Q(k)y(k) + u'(k)R(k)u(k) \tag{3.23}$$

with a moving terminal constraint $x(i+N) = 0^{\dagger}$. It will be interesting to investigate some relationships between the modified control law (3.11) and the fixed terminal control law (3.1) from which the control law (3.11) is obtained. We can show that the quadratic cost for the system (1.1)-(1.2) with the control law (3.11) is no more than that of the fixed terminal control (3.1) with $i_f = i_o + N$.

<u>Theorem 3.3.</u> The quadratic cost (1.3) for the system (1.1)-(1.2) with the control (3.11) has the following bounds:

$$x'(i_o)K(i_o,i_1)x(i_o) \leq \sum_{i_o}^{i_1-1} y'(i)Q(i)y(i) + u'(i)R(i)u(i)$$

$$\leq x'(i_o)P^{-1}(i_o,i_o+N)x(i_o). \tag{3.24}$$

Proof: We have the following inequality:

$$x'(i)P^{-1}(i,i+N)x(i) - x'(i+1)P^{-1}(i+1,i+1+N)x(i+1)$$

$$= x'(i)P^{-1}(i,i+1+N)x(i) - x'(i+1)P^{-1}(i+1,i+1+N)x(i+1) +$$

$$x(i)[P^{-1}(i,i+N) - P^{-1}(i,i+1+N)]x(i)$$

$$\geq x'(i)P^{-1}(i,i+1+N)x(i) - x'(i+1)P^{-1}(i+1,i+1+N)x(i+1) \tag{3.25}$$

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†Although the control (3.11) is optimal for the above problem, our main thesis is that the receding horizon concept leads to one of the easiest stabilization methods for linear systems.

[illegible]

$$[illegible] \qquad (3.[illegible])$$

[illegible]

$$[illegible]$$

$$[illegible] \qquad (3.[illegible])$$

$$[illegible]$$

Replacing j by i+1+N in (3.27) and combining (3.26) and (3.2[illegible]) we obtain

$$x'(i)P^{-1}(i,i+1+N)x(i) - x'(i+1)P^{-1}(i+1,i+1+N)x(i+1)$$
$$\geq x'(i)C'(i)Q(i)C(i)x(i) + u'(i)R(i)u(i), \qquad (3.28)$$

from which follows that

$$\sum_{i_o}^{i_1-1} y'(i)Q(i)y(i) + u'(i)R(i)u(i)$$
$$\leq x'(i_o)P^{-1}(i_o,i_o+N)x(i_o) - x'(i_1)P^{-1}(i_1,i_1+N)x(i_1).$$

The lower bound is obvious. This completes the proof.

When the control law (3.11) is used as a suboptimal control to the steady state control for (1.11), the error bound is given in Theorem 3.3. The stabilization of the system (1.1) with a prescribed degree of stability can be obtained from the result in Theorem 3.2.

which follows from the fact that $P^{-1}(i,i+N) \geq P^{-1}(i,i+1+N)$ as given in Lemma 3.1(2). From (1.5) and (1.6) we can obtain

$$x'(i)p(i) - x'(i+1)p(i+1) = x'(i)C'(i)Q(i)C(i)x(i) + u'(i)R(i)u(i) \tag{3.26}$$

for both the free and fixed terminal conditions (cf. [12]). For the fixed terminal condition $x(j) = 0$, the quantities $p(i)$, $u(i)$ and $x(i+1)$ in (3.26) are given by

$$p(i) = P^{-1}(i,j)x(i)$$

$$u(i) = -R^{-1}(i)B'(i)[B(i)R^{-1}(i)B'(i)+P(i+1,j)]^{-1}\Phi(i+1,i)x(i) \tag{3.27}$$

$$x(i+1) = [\Phi(i+1,i)-B(i)R^{-1}(i)B'(i)[B(i)R^{-1}(i)B'(i)+P(i+1,j)]^{-1}\Phi(i+1,i)]x(i).$$

Replacing j by i+1+N in (3.27) and combining (3.26) and (3.25) we obtain

$$\begin{aligned} &x'(i)P^{-1}(i,i+1+N)x(i) - x'(i+1)P^{-1}(i+1,i+1+N)x(i+1) \\ &\qquad \geq x'(i)C'(i)Q(i)C(i)x(i) + u'(i)R(i)u(i), \end{aligned} \tag{3.28}$$

from which follows that

$$\begin{aligned} &\sum_{i_o}^{i_1-1} y'(i)Q(i)y(i) + u'(i)R(i)u(i) \\ &\qquad \leq x'(i_o)P^{-1}(i_o,i_o+N)x(i_o) - x'(i_1)P^{-1}(i_1,i_1+N)x(i_1). \end{aligned}$$

The lower bound is obvious. This completes the proof.

When the control law (3.11) is used as a suboptimal control to the steady state control for (1.11), the error bound is given in Theorem 3.3. The stabilization of the system (1.1) with a prescribed degree of stability can be obtained from the result in Theorem 3.2.

<u>Theorem 3.4</u>. Assume that the conditions in Theorem 3.2 hold. Then the system (1.1) with the feedback control law

$$u(i) = -R^{-1}(i)B'(i)\hat{P}_{\alpha}^{-1}(i+1,i+1+N)\Phi(i+1,i)x(i), \ \alpha \geq 1 \tag{3.29}$$

is uniformly asymptotically stable, where $\hat{P}_{\alpha}(i+1,i+1+N)$ is obtained from (3.2) with $\Phi(i+1,i)$ replaced by $\alpha\Phi(i+1,i)$. Furthermore, there exists an $L > 0$ and a γ satisfying $0 < \gamma < 1$ such that the transition matrix $\Phi_{P_\alpha}(i,i_o)$ of the closed-loop system (1.1)-(3.29) satisfies

$$|| \Phi_{P_\alpha}(i,i_o) || \leq L\left(\frac{\gamma}{\alpha}\right)^{(i-i_o)}, \quad i_o \leq i < \infty . \tag{3.30}$$

Proof: Consider the system

$$\hat{x}(i+1) = \alpha\Phi(i+1,i)\hat{x}(i) + B(i)\hat{u}(i) \tag{3.31}$$

$$\hat{y}(i) = C(i)\hat{x}(i) . \tag{3.32}$$

It is easily seen that the pairs $\{\alpha\Phi(i+1,i), B(i)\}$ and $\{\alpha\Phi(i+1,i), C(i)\}$ are uniformly completely controllable and observable, respectively, if $\{\Phi(i+1,i), B(i)\}$ and $\{\Phi(i+1,i), C(i)\}$ are uniformly conpletely controllable and observable respectively. Thus $P_\alpha(i+1,i+1+N)$ satisfies the properties in Lemma 3.1. Let $\hat{\Phi}_{P\alpha}(i,i_o)$ be the state transition matrix of the system (3.31) with the control law

$$\hat{u}(i) = -R^{-1}(i)B'(i)\hat{P}_{\alpha}^{-1}(i+1,i+1+N)\alpha\Phi(i+1,i)x(i) .$$

Then it is easy to see that $\hat{\Phi}_{P\alpha}(i,i_o) = \alpha^{(i-i_o)}\Phi_{P\alpha}(i,i_o)$. From Theorem 3.2 it follows that $||\hat{\Phi}_{P\alpha}(i,i_o)|| \leq L\gamma^{(i-i_o)}$ for some $L > 0$ and $0 < \gamma < 1$. From this it follows that $||\Phi_{P_\alpha}(i,i_o)|| \leq L\left(\frac{\gamma}{\alpha}\right)^{(i-i_o)}$. This completes the proof.

In the next section we can obtain the corresponding results for linear time invariant systems with simplified forms. Especially, the feedback gain is shown to be a constant matrix.

IV. Linear Time Invariant Systems

Consider a linear time invariant system

$$x(i+1) = \Phi x(i) + Bu(i) \tag{4.1}$$

$$y(i) = Cx(i) \tag{4.2}$$

where $\{\Phi, B, C\}$ are constant matrices and Φ is nonsingular, together with a cost function

$$\sum_{i_o}^{i_f-1} y'(i)Qy(i) + u'(i)Ru(i) \tag{4.3}$$

where $Q = D'D \geq 0$ and $R > 0$ are constant weighting matrices. If the pair $\{\Phi,B\}$ is completely controllable, then the minimization of (4.3) subject to the end point constraint $x(i_f) = 0$ leads to the optimal feedback control law (cf.(3.1))

$$u(i) = -R^{-1}B'\hat{P}^{-1}(i_f-i)\Phi x(i) \tag{4.4}$$

if the inverse in (4.4) exists, where $\hat{P}(k)$ is obtained from

$$\hat{P}(k+1) = \Phi^{-1}\hat{P}(k)\Phi'^{-1} - \Phi^{-1}\hat{P}(k)\Phi'^{-1}C'D'(I+DC\Phi^{-1}\hat{P}(k)\Phi'^{-1}C'D')^{-1}DC\Phi^{-1}\hat{P}(k)\Phi'^{-1}$$
$$+ BR^{-1}B', \hat{P}(0) = 0.$$

The result analogous to Theorem 3.2 for this case is contained in the following.

<u>Theorem 4.1</u>. Suppose Φ is nonsingular. If the pair $\{\Phi,B\}$ is completely controllable, then for a fixed integer N satisfying $n \leq N < \infty$ the system (4.1) is asymptotically stable with the fixed gain feedback control law

$$u(i) = -R^{-1}B'\hat{P}^{-1}(N)\Phi x(i) \tag{4.6}$$

where $\hat{P}(N)$ can be obtained from (4.5) corresponding to any chosen pair $\{Q,R\}$ with $Q \geq 0$ and $R > 0$. With the additional condition that the pair $\{\Phi,C\}$ is detectable and $Q > 0$, the result holds with $n \leq N \leq \infty$.

Proof: Although the proof is merely a specialization of Theorem 3.2 to the time invariant case, it is noted that a direct proof of asymptotic stability can be given in this case using the Lyapunov function $V(\hat{x}) = \hat{x}'\Phi^{-1}\hat{P}(N)\Phi'^{-1}\hat{x}$ for the system $\hat{x}(i+1) = [\Phi - BR^{-1}B'\hat{P}^{-1}(N)\Phi]'\hat{x}(i)$.

The control law (4.6) is a generalization of a stabilizing feedback control law given in [5] involving the inverse of the controllability Gramian in that the result in [5] is obtained by choosing $Q = 0$ (or $C = 0$) in (4.5). That is, with $Q = 0$, $\hat{P}(N)$ is given by

$$\hat{P}(N) = \sum_{i=0}^{N-1} \Phi^{-i}BR^{-1}B'\Phi'^{-i}$$

and

$$\hat{P}^{-1}(N) = \Phi'^{N-1}\left[\sum_{i=0}^{N-1} \Phi^{i}BR^{-1}B'\Phi'^{i}\right]^{-1}\Phi^{N-1}. \tag{4.7}$$

The advantage of the control law (4.6) to the one employing (4.7) is that the former can weight the state, or the output, by choosing a proper Q. In [13], it is shown that the matrix Φ in (4.7) can be singular for a controllable single input system. The control (4.6) can be regarded as a generalization of the stable steady state control of (1.11) as shown in Fig. 2. From the special structure of the time invariant system, the condition of Theorem 4.1 can be weakened as follows:

Proposition 4.1. Assume Φ is nonsingular. If the pair $\{\Phi,B\}$ is stabilizable, then the system (4.1) is asymptotically stable with the following control law

$$u(i) = -R^{-1}B'\hat{P}^{\dagger}(N)\Phi x(i), \quad N \geq n, \tag{4.8}$$

where $\hat{P}^{\dagger}(N)$ is the generalized inverse of the matrix $\hat{P}(N)$ obtained from (4.5) for any $Q \geq 0$ and $R > 0$.

Proof: If $\{\Phi,B\}$ is stabilizable, then there exists a nonsingular real matrix S such that with the transformation $\tilde{x}(i) = Sx(i)$, the system (4.1)-(4.8) is transformed to

$$\tilde{x}(i+1) = [\tilde{\Phi} - \tilde{B}R^{-1}\tilde{B}'\tilde{P}^{\dagger}(N)\tilde{\Phi}]\tilde{x}(i) \ , \quad N > n,$$

where $\tilde{P}(N)$ is obtained from

$$\tilde{P}(k+1) = \tilde{\Phi}^{-1}\tilde{P}(k)\tilde{\Phi}^{-1} - \tilde{\Phi}^{-1}\tilde{P}(k)\tilde{\Phi}'^{-1}S'^{-1}C'D'(I+DCS^{-1}\tilde{\Phi}^{-1}\tilde{P}(k)$$
$$\times \tilde{\Phi}'^{-1}S'^{-1}C'D')^{-1}DCS^{-1}\tilde{\Phi}^{-1}P(k)\tilde{\Phi}'^{-1} + \tilde{B}R^{-1}\tilde{B}'$$

$$\tilde{P}(0) = 0$$

and

$$\tilde{\Phi} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \text{and } DCS^{-1} = \begin{bmatrix} H_1 \ , \ H_2 \end{bmatrix}$$

where Φ_{11} and Φ_{22} are nonsingular, $\{\Phi_{11},B_1\}$ is controllable, and Φ_{22}, is a stable matrix. Direct computation yields

$$\tilde{x}(i+1) = \begin{bmatrix} \Phi_{11} - B_1R^{-1}B_1'\tilde{P}_{11}^{-1}(N)\Phi_{11} \ , & \Phi_{12} \\ 0 \ , & \Phi_{22} \end{bmatrix} \tilde{x}(i) \qquad (4.9)$$

where $\tilde{P}_{11}(k)$ satisfies (4.5) with Φ,DC, and B replaced by Φ_{11}, H_1, and B_1 respectively. Thus the matrix in (4.9) is a stable matrix from Theorem 4.1. This completes the proof.

The result in Theorem 3.4 can be restated as follows: If the pair $\{\Phi,B\}$ is completely controllable and Φ nonsingular, then for a fixed N satisfying $n \leq N < \infty$ the system (4.1) is asymptotically stable with the control law

$$u(i) = -R^{-1}B'\hat{P}_{\alpha}^{-1}(N)\Phi x(i), \qquad \alpha \geq 1 \tag{4.10}$$

where $\hat{P}_{\alpha}(N)$ is obtained from (4.5) with Φ replaced by $\alpha\Phi$. Furthermore, all poles of the closed-loop system are located inside the circle of radius $|s| = \frac{1}{\alpha}$ in the complex plane.

The results for the above regulator problem have other implications in a dual problem of filtering theory.

V. Application to a Dual Problem in Filtering Theory.

Consider a linear stochastic system with white noises,

$$\begin{aligned} x(i+1) &= \Phi(i+1,i)x(i) + B(i)w(i) \\ y(i) &= C(i)x(i) + v(i) \end{aligned} \tag{5.1}$$

where $Ex(i_o) = \bar{x}(i_o)$, $E(x(i_o) - \bar{x}(i_o))(x(i_o) - \bar{x}(i_o))' = \sum_o$, $Ew(i) = Ev(i) = 0$, $Ex(i_o)w'(i_o) = Ew(i)v'(i) = 0$, $Ew(i)w'(i) = Q(i)$, and $Ev(i)v'(i) = R(i)$. The standard Kalman filter solution is given by

$$\begin{aligned} &\hat{x}(i+1) = \Phi_i\{\hat{x}(i) + \sum(i_o,i)C'(i)[C(i)\sum(i_o,i)C'(i)+R(i)]^{-1}[y(i)-C(i)\hat{x}(i)]\} \\ &\hat{x}(i_o) = \bar{x}(i_o) \end{aligned} \tag{5.2}$$

where $\sum(i_o,i)$ is obtained from

$$\begin{aligned} \sum(i,j+1) = {} & \Phi_i\sum(i,j)\{I-C'(i)[C(i)\sum(i_o,i)C'(i)+R(i)]^{-1}C(i)\sum(i,j)\}\Phi_i' \\ & + B(i)Q(i)B'(i), \qquad \sum(i,i) = \sum_o, \quad i \leq j. \end{aligned} \tag{5.3}$$

The filtering error $e(i) \triangleq \hat{x}(i) - x(i)$ has the mean $\bar{e}(i)$ represented by

$$\bar{e}(i+1) = \Phi_i\{I - \sum(i_o,i)C'(i)[C(i)\sum(i_o,i)C'(i) + R(i)]^{-1}C(i)\}\bar{e}(i) \tag{5.4}$$

and variance given by $\sum(i_o,i)$. The dual problem of the fixed terminal minimum energy problem, (1.1) to (1.4), is considered as the standard filtering problem with a completely unknown initial condition, i.e.,

$$Ex(i_o) = \text{unknown and } \sum_o = \infty. \tag{5.5}$$

Thus the filtering problem with a completely unknown initial condition is given by

$$\hat{x}(i+1) = \Phi_i\{\hat{x}(i)+\Gamma^{-1}(i_o,i)C'(i)[C(i)\Gamma^{-1}(i_o,i)C'(i)+R(i)]^{-1}[y(i)-C(i)\hat{x}(i)]\} \quad (5.6)$$

$\hat{x}(i_o)$ = arbitrary,

where

$$\Gamma(i,j+1) = \Phi_i'^{-1}\hat{\Gamma}(i,j)\Phi_i^{-1}\{I+C'(i)Q(i)C(i)\Phi_i^{-1}\hat{\Gamma}(i,j)\Phi_i^{-1}\}^{-1}$$

$$\hat{\Gamma}(i,j) \triangleq \Gamma(i,j) + C'(i)R^{-1}(i)C(i) \quad (5.7)$$

$$\Gamma(i,i) = 0 \quad ,$$

if the inverse in (5.6) exists. The error mean equation for the estimator (5.6) is given by

$$\bar{e}(i+1) = \Phi_i\{I-\hat{\Gamma}^{-1}(i_o,i)C'(i)R^{-1}(i)C(i)\}\bar{e}(i) \quad . \quad (5.8)$$

Uniform complete controllability and observability of the system (5.1) are defined as usual with the dual system of (5.1). We state the corresponding results in this case without proof.

Corollary 5.1. Assume $R(i)$ satisfies (1.18) and $0 \le Q(i) \le \alpha_5 I$. If the pair $\{\Phi(i+1,i) , C(i)\}$ is uniformly completely observable and $B(i)$ is bounded, then for a fixed N satisfying $\ell_o + 1 \le N < \infty$ the state estimator

$$\hat{x}(i+1) = \Phi(i+1,i)\hat{x}(i) + \Phi(i+1,i)\Gamma^{-1}(i-N,i)C'(i)[C(i)\Gamma^{-1}(i-N,i)C'(i) + R(i)]^{-1}[y(i) - C(i)\hat{x}(i)] \quad (5.9)$$

is uniformly asymptotically stable, where $\Gamma(i,j)$ is obtained from (5.7).

In Corollary 5.1 the value of N can be infinite under the additional assumption of uniform complete controllability of the pair $\{\Phi(i+1,i),B(i)\}$ and $\alpha_4 I \le Q(i) \le \alpha_5 I$. It is easily seen that the estimator (5.9) is the optimal estimator which minimizes the criterion $E(x(i+1) - \hat{x}(i+1))'(x(i+1) - \hat{x}(i+1))$ based on the moving information $\{y(i-N), \ldots y(i)\}$ and a completely unknown moving initial condition $E(x(i-N) - \bar{x}(i-N))(x(i-N) - \bar{x}(i-N))' = \infty$. It is also noted that the significance of the estimator (5.6) lies in the fact that it is one of the easiest ways to obtain a linear stable estimator.

VI. Concluding Remarks

An advantage of the control law (3.11) is that the stabilizing feedback gains are obtained by summing a Riccati equation backward in time over a finite interval, rather than an infinite time interval. The control law (4.6) for time invariant systems generalizes a well known method of feedback stabilization due to Kleinman, and can be interpreted as providing a means for weighting the state or the output in the cost function by choosing $Q \neq 0$. In the case of time invariant systems, the modified control law (4.6) can also be interpreted as a practical way to avoid the singularity near the terminal time of the optimal control (4.4) when the argument $(i_f - i)$ in $\hat{P}^{-1}(i_f - i)$ is frozen at some time $N = i_f - i \geq n$. The important consideration is that such expediency still renders the resulting feedback control law asymptotically stable. Similar consideration applies to the comparison between the control laws (3.1) and (3.11) which pertain to the time varying system (1.1)-(1.2). Corresponding results for a dual problem in filtering theory have also been discussed in this paper.

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Appendix

A. Proof of Theorem 2.1

The proof is almost the same as in [8] for continuous time systems. Thus we only sketch its proof. Vector functions of time, $u(k)$ and $y(k)$, $k\epsilon[i_o,i_f-1]$ can be considered as elements of the Hilbert space of product spaces $H_1 = R^m \times R^m \times .. \times R^m$ and $H_2 = R^p \times R^p \times .. \times R^p$ (both $i_f - i_o$ times) with corresponding inner products defined as $<u_1,u_2>_R = \sum_{k=i_o}^{i_f-1} u_1'(k)R(k)u_2(k)$ and $<y_1,y_2>_Q = \sum_{k=i_o}^{i_f-1} y_1'(k)Q(k)y_2(k)$. The norms for the above spaces are induced by these inner products. The operator G defined in (2.3) maps H_1 into H_2. Let $y_h(i) = C(i)\Phi(i,i_o)x_o$ and $J_h = |y_h|_Q^2$. Let $\hat{u}$ be an arbitrary control and $\hat{y}$ the corresponding output. Then it holds that $\hat{y} = y_h + G\hat{u}$ and $J(\hat{u},x_o) = |\hat{u}|_R^2 + |G\hat{u}+y_h|_Q^2$. The vector pair $(\hat{u},\hat{y})$ of the Hilbert space $H_3 = H_1 \times H_2$ with inner product $<(\hat{u}_1,\hat{y}_1), (\hat{u}_2,\hat{y}_2)> = <\hat{u}_1,\hat{u}_2>_R + <\hat{y}_1,\hat{y}_2>_Q$ belongs to the linear variety V of H_3. By the projection theorem, $<(u^*,y^*), (u^*-\hat{u},y^*-\hat{y})> = 0$, from which follows that $J(u^*,x_o) \triangleq J^* = <(u^*,u^*), (y^*,y^*)> = <u^*,\hat{u}>_R + <y^*,\hat{y}>_Q$. Also from the fact that $J^* = |u^*|_R^2 + |y_h + Gu^*|_Q^2$, we can obtain $J^* = J_h - |Gu^*|_Q^2 - |u^*|_R^2 = J_h + <Gu^*,y_h>_Q$. From this equation follows that $|Gu^*|_Q^2 \geq (J_h - J^*)^2/J_h$ and $J^*/J_h \leq 1 - (|Gu^*|^2/J_h)(1 + 1/||G||^2)$. Thus $1 \geq J^*/J_h \geq 1/(1 + ||G||^2)$. It is known that $J^* = x_o'K(i_o,i_f)x_o$ and $J_h = x_o'N(i_o,i_f)x_o$, which together with the above inequality verifies (2.1).

B. Proof of Theorem 2.2

The proof is almost the same as in [6,7] and thus a sketch will suffice. From (2.4), uniform complete controllability implies that the system (1.1) is bounded. It is easily seen that the bounded system (1.1) is uniformly completely controllable if, and only if, there exists a positive integer $\ell_c > 0$ such that $W(i,i+\ell_c) \geq \alpha_1 I$ for all i. Then it can be shown by contradiction that a bounded system is uniformly completely controllable if, and only if, there exists a $\ell_c > 0$ such that for every state $\zeta \in R^n$ and for any time i, there exists an input $\hat{u}$ defined on $[i,i+l_c-1]$ such that if $x(i) = \zeta$ then $x(i+\ell_c) = 0$ and $|\hat{u}(k)| \leq \gamma(|\zeta|)$ for all $k \in [i,i+\ell_c-1]$. It is readily verified that if the control $v(k) = \hat{u}(k) - K(k)\hat{x}(k)$ is the input to the feedback closed-loop system $z(i+1) = [\Phi(i+1,i)+B(i)K(i)]z(i) + B(i)v(i)$ where $\hat{x}(k)$ is the trajectory of the open-loop system due to the minimum energy control $\hat{u}$, then $z(i) = \zeta$ and $z(i+\ell_c) = 0$ (in fact $z(k) = x(k)$ for all $k \in [i,i+\ell_c])$.

Thus we have $|v(k)| \leq |\hat{u}(k)| + ||K(k)|| |\hat{x}(k)| \leq \gamma(|\zeta|) +$

$\alpha_{12}|\Phi(k,i)\zeta + \sum_{j=i}^{k} \Phi(k,j)B(j)\hat{u}(j)| \leq \tilde{\gamma}(|\zeta|)$ for all $k \in [i,i+\ell_c-1]$ where $\gamma(.)$ and $\tilde{\gamma}(.)$ are bounded functions on bounded intervals.

The last inequality of the above relation follows from (1.14) and (2.4).

C. Proof of Lemma 3.1

(1). The matrix $\hat{P}(i,j)$ defined in (3.1) can be obtained from the following free terminal problem:

$$\hat{x}(i+1) = \Phi'^{-1}(i+1,i)\hat{x}(i) + \Phi'^{-1}(i+1,i)C'(i)D'(i)\hat{u}(i)$$

$$\hat{y}(i) = B'(i-1)\hat{x}(i) \qquad \hat{x}(i_o) = \hat{x}_o$$

with the cost function

$$\sum_{k=i_o}^{j-1} \hat{y}'(k)R^{-1}(k-1)\hat{y}(k) + \hat{u}'(k)\hat{u}(k) \quad .$$

Since $\hat{x}_o'\hat{P}(i_o,j)\hat{x}_o$ is the cost of the above problem, it is clear that $\hat{P}(i,j_1) \leq \hat{P}(i,j_2)$.

(2). Since $x_o'P^{-1}(i_o,j)x_o$ is the minimum cost of the fixed terminal problem, the result is straightforward.

(3). The upper bound is obvious from (1.14) and (2.4). Since $\hat{P}(i,j)$ comes from the free terminal problem given in (1), we have from Theorem 2.1

$$\hat{P}(i,i+N) \geq \frac{1}{1 + ||G||} W(i,i+N)$$

where

$$(G\hat{u})(k) = \sum_{j=i}^{k-1} R^{-1/2}(k-1)B'(k-1)\Phi'^{-1}(k,j)C'(j)D'(j)\hat{u}(j), \; k\epsilon[i,i+N-1] \quad .$$

Thus we have

$$|(G\hat{u})(k)| \leq \sum_{j=i}^{k-1} ||R^{-1/2}(k-1)B'(k-1)\Phi'^{-1}(k,j)C'(j)D'(j)|| \; |\hat{u}(j)|$$

$$\leq \sum_{j=i}^{k-1} \alpha_6^{-1/2} \alpha_{10} \left[\sup_{1\leq \ell \leq N-1} \alpha_3(\ell)\right] \alpha_{13}\alpha_4^{1/2}|\hat{u}(j)|$$

which yields that $|G\hat{u}| \leq \sqrt{N-1}\,\alpha_6^{-1/2}\,\alpha_{10}[\sup_{1\leq \ell \leq n-1} \alpha_3(\ell)]\alpha_{13}\alpha_4^{1/2}|\hat{u}|$

and $||G||^2 \leq (N-1)\alpha_6^{-1}\alpha_{10}^2[\sup_{1\leq \ell \leq N-1} \alpha_3(\ell)]^2\,\alpha_{13}^2\alpha_4$.

(4). The proof may be found in its dual form in [2,3].

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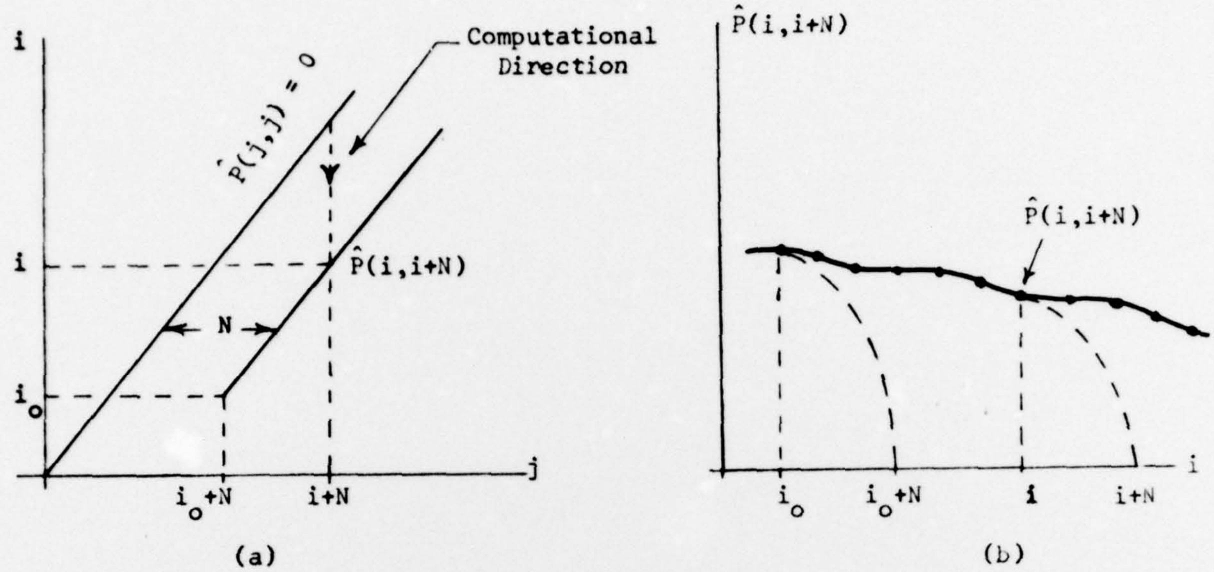

Fig. 1 Characterization of $\hat{P}(i,j)$

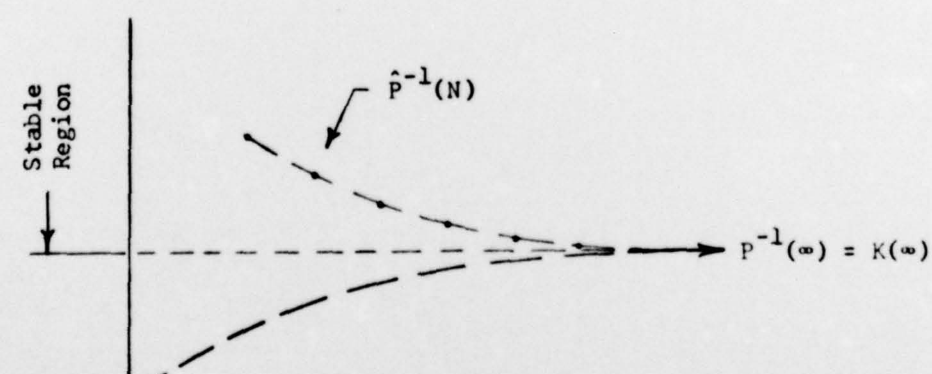

Fig. 2 Comparison Between $\hat{P}^{-1}(N)$ and $K(N)$

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This report considers a feedback control law for linear time-varying and time invariant discrete systems based on a receding horizon concept applied to a minimum energy problem with fixed terminal constraints. The control law is shown to be asymptotically stable and to result in a new method for stabilizing

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20. Abstract (continued)

linear time-varying systems as well as extending some well known methods for stabilizing time invariant systems. In particular, the stabilizing gains of the feedback control law are obtained from the solution to a discrete Riccati equation over an arbitrary finite time interval, which is relatively easy to compute. The gain matrix reduces to a constant matrix for linear time invariant systems. Some stability results ~~in [2] and [5]~~ will turn out to be special cases of these results. The results parallel those ~~of [4]~~ for linear continuous time systems, although the technical details are tedious and more involved.